

On the Intersection of All Critical Sets of a Unicyclic Graph

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Abstract

A set $S \subseteq V$ is *independent* in a graph $G = (V, E)$ if no two vertices from S are adjacent. The *independence number* $\alpha(G)$ is the cardinality of a maximum independent set, while $\mu(G)$ is the size of a maximum matching in G . If $\alpha(G) + \mu(G) = |V|$, then G is called a *König-Egerváry graph*. The number $d_c(G) = \max\{|A| - |N(A)| : A \subseteq V\}$ is called the *critical difference* of G [21], where $N(A) = \{v : v \in V, N(v) \cap A \neq \emptyset\}$. By $\text{core}(G)$ ($\text{corona}(G)$) we denote the intersection (union, respectively) of all maximum independent sets, while by $\text{ker}(G)$ we mean the intersection of all critical independent sets. A connected graph having only one cycle is called *unicyclic*.

It is known that the relation $\text{ker}(G) \subseteq \text{core}(G)$ holds for every graph G [13], while the equality is true for bipartite graphs [14]. For König-Egerváry unicyclic graphs, the difference $|\text{core}(G)| - |\text{ker}(G)|$ may equal any non-negative integer.

In this paper we prove that if G is a non-König-Egerváry unicyclic graph, then: (i) $\text{ker}(G) = \text{core}(G)$ and (ii) $|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G) + 1$. Pay attention that $|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G)$ holds for every König-Egerváry graph [14].

Keywords: maximum independent set, core, corona, matching, critical set, unicyclic graph, König-Egerváry graph.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X . By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. For $F \subset E(G)$, by $G - F$ we denote the partial subgraph of G obtained by deleting the edges of F , and we use $G - e$, if $W = \{e\}$. If $A, B \subset V$ and $A \cap B = \emptyset$, then (A, B) stands for the set $\{e = ab : a \in A, b \in B, e \in E\}$. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, and

$N(A) = \cup\{N(v) : v \in A\}$, $N[A] = A \cup N(A)$ for $A \subset V$. By C_n, K_n we mean the chordless cycle on $n \geq 4$ vertices, and respectively the complete graph on $n \geq 1$ vertices.

A set S of vertices is *independent* if no two vertices from S are adjacent, and an independent set of maximum size will be referred to as a *maximum independent set*. The *independence number* of G , denoted by $\alpha(G)$, is the cardinality of a maximum independent set of G .

Let $\text{core}(G) = \cap\{S : S \in \Omega(G)\}$ [9], and $\text{corona}(G) = \cup\{S : S \in \Omega(G)\}$ [2], where $\Omega(G) = \{S : S \text{ is a maximum independent set of } G\}$.

Theorem 1.1 [2] *For every $S \in \Omega(G)$, there is a matching from $S - \text{core}(G)$ into $\text{corona}(G) - S$.*

An edge $e \in E(G)$ is α -critical whenever $\alpha(G - e) > \alpha(G)$. Notice that $\alpha(G) \leq \alpha(G - e) \leq \alpha(G) + 1$ holds for each edge e .

The number $d(X) = |X| - |N(X)|$, $X \subseteq V(G)$, is called the *difference* of the set X , while $d_c(G) = \max\{d(X) : X \subseteq V\}$ is called the *critical difference* of G . A set $U \subseteq V(G)$ is *critical* if $d(U) = d_c(G)$ [21]. The number $\text{id}_c(G) = \max\{d(I) : I \in \text{Ind}(G)\}$ is called the *critical independence difference* of G . If $A \subseteq V(G)$ is independent and $d(A) = \text{id}_c(G)$, then A is called a *critical independent set* [21]. Clearly, $d_c(G) \geq \text{id}_c(G)$ is true for every graph G .

Theorem 1.2 [21] *The equality $d_c(G) = \text{id}_c(G)$ holds for every graph G .*

For a graph G , let denote $\text{ker}(G) = \cap\{S : S \text{ is a critical independent set}\}$.

Theorem 1.3 *If G is a graph, then*

- (i) [13] $\text{ker}(G)$ is a critical independent set and $\text{ker}(G) \subseteq \text{core}(G)$;
- (ii) [14] $\text{ker}(G) = \text{core}(G)$, whenever G is bipartite.

A matching (i.e., a set of non-incident edges of G) of maximum cardinality $\mu(G)$ is a *maximum matching*, and a *perfect matching* is one covering all vertices of G . An edge $e \in E(G)$ is μ -critical provided $\mu(G - e) < \mu(G)$.

It is well-known that $\lfloor n/2 \rfloor + 1 \leq \alpha(G) + \mu(G) \leq n$ hold for any graph G with n vertices. If $\alpha(G) + \mu(G) = n$, then G is called a *König-Egerváry graph* [3], [18]. Several properties of König-Egerváry graphs are presented in [8], [10], [12].

According to a celebrated result of König, [7], and Egerváry, [5], any bipartite graph is a König-Egerváry graph. This class includes also non-bipartite graphs (see, for instance, the graph G in Figure 1).

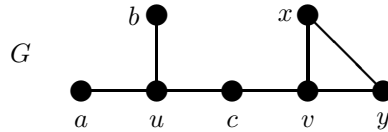


Figure 1: A König-Egerváry graph with $\alpha(G) = |\{a, b, c, x\}|$ and $\mu(G) = |\{au, cv, xy\}|$.

Theorem 1.4 [10] *If G is a König-Egerváry graph, then every maximum matching matches $N(\text{core}(G))$ into $\text{core}(G)$.*

The graph G is called *unicyclic* if it is connected and has a unique cycle, which we denote by $C = (V(C), E(C))$. Let

$$N_1(C) = \{v : v \in V(G) - V(C), N(v) \cap V(C) \neq \emptyset\},$$

and $T_x = (V_x, E_x)$ be the tree of $G - xy$ containing x , where $x \in N_1(C)$, $y \in V(C)$.

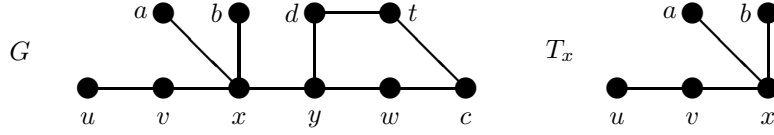


Figure 2: G is a unicyclic non-König-Egerváry graph with $V(C) = \{y, d, t, c, w\}$.

The following result shows that a unicyclic graph is either a König-Egerváry graph or each edge of its cycle is α -critical.

Lemma 1.5 [16] *If G is a unicyclic graph of order n , then*

- (i) $n - 1 \leq \alpha(G) + \mu(G) \leq n$;
- (ii) $n - 1 = \alpha(G) + \mu(G)$ if and only if each edge of the unique cycle is α -critical.

Theorem 1.6 [16] *Let G be a unicyclic non-König-Egerváry graph. Then the following assertions are true:*

- (i) each $W \in \Omega(T_x)$ can be enlarged to some $S \in \Omega(G)$;
- (ii) $S \cap V(T_x) \in \Omega(T_x)$ for every $S \in \Omega(G)$;
- (iii) $\text{core}(G) = \cup \{\text{core}(T_x) : x \in N_1(C)\}$.

Unicyclic graphs keep enjoying plenty of interest, as one can see, for instance, in [1], [4], [6], [11], [17], [19], [20].

In this paper we analyze the relationship between several parameters of a unicyclic graph G , namely, $\text{core}(G)$, $\text{corona}(G)$, $\text{ker}(G)$.

2 Results

Lemma 2.1 *If G is a unicyclic non-König-Egerváry graph, then*

- (i) $\text{core}(G) \cap N[V(C)] = \emptyset$;
- (ii) there exists a matching from $N(\text{core}(G))$ into $\text{core}(G)$.

Proof. (i) Let $ab \in E(C)$. By Lemma 1.5(ii), the edge ab is α -critical. Hence there are $S_a, S_b \in \Omega(G)$, such that $a \in S_a$ and $b \in S_b$. Since $a \notin S_b$, it follows that $a \notin \text{core}(G)$, and because $a \in S_a$, we infer that $N(a) \cap \text{core}(G) = \emptyset$. Consequently, we obtain that $\text{core}(G) \cap N[V(C)] = \emptyset$.

(ii) If $\text{core}(G) = \emptyset$, then the conclusion is clear.

Assume that $\text{core}(G) \neq \emptyset$. By Theorem 1.4, in each tree T_x there is a matching M_x from $N(\text{core}(T_x))$ into $\text{core}(T_x)$. By part (i), we have that $V(C) \cap N[\text{core}(G)] = \emptyset$. Taking into account Theorem 1.6(ii), we see that the union of all these matchings M_x gives a matching from $N(\text{core}(G))$ into $\text{core}(G)$. ■

It is worth mentioning that the assertion in Lemma 2.1(ii) is true for every König-Egerváry graph, by Theorem 1.4. The graph G_2 from Figure 3 shows that Lemma 2.1(i) may fail for unicyclic König-Egerváry graphs.

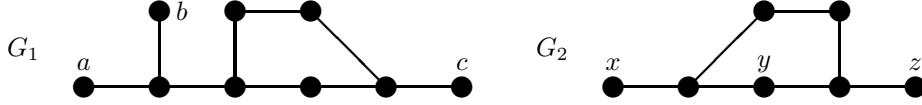


Figure 3: König-Egerváry graphs with $\text{core}(G_1) = \{a, b, c\}$ and $\text{core}(G_2) = \{x, y, z\}$.

Theorem 2.2 *If G is a König-Egerváry graph, then*

- (i) [10] $N(\text{core}(G)) = \cap \{V(G) - S : S \in (G)\}$, i.e., $N(\text{core}(G)) = V(G) - \text{corona}(G)$;
- (ii) [14] $|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G)$.

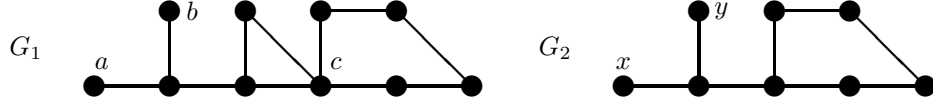


Figure 4: Non-König-Egerváry graphs with $\text{core}(G_1) = \{a, b\}$ and $\text{core}(G_2) = \{x, y\}$.

The graphs G_1, G_2 from Figure 4 satisfy $\text{corona}(G_1) \cup N(\text{core}(G_1)) = V(G_1) - \{c\}$, while $\text{corona}(G_2) \cup N(\text{core}(G_2)) = V(G_2)$.

Theorem 2.3 *If G is unicyclic non-König-Egerváry graph, then*

$$\begin{aligned} \text{corona}(G) \cup N(\text{core}(G)) &= V(G), \\ \text{corona}(G) &= V(C) \cup (\cup \{\text{corona}(T_x) : x \in N_1(C)\}). \end{aligned}$$

Proof. For the first equality, it is enough to show that $V(G) \subseteq \text{corona}(G) \cup N(\text{core}(G))$.

Let $a \in V(G)$.

Case 1. $a \in V(C)$. If $b \in N(a) \cap V(C)$, then, by Lemma 1.5(ii), the edge ab is α -critical. Hence $a \in \text{corona}(G)$.

Case 2. $a \in V(G) - V(C)$. Since $V(G) = V(C) \cup (\cup \{V(T_x) : x \in N_1(C)\})$, it follows that there is some $y \in N_1(C)$, such that $a \in V(T_y)$. According to Theorem 2.2(i), we know that $V(T_y) = \text{corona}(T_y) \cup N(\text{core}(T_y))$.

By Theorem 1.6(i), $\text{corona}(T_x) \subseteq \text{corona}(G)$ for every $x \in N_1(C)$. Therefore, either $a \in \text{corona}(T_y) \subseteq \text{corona}(G)$, or $a \in N(\text{core}(T_y)) \subseteq N(\text{core}(G))$, because $\text{core}(T_y) \subseteq \text{core}(G)$ in accordance with Theorem 1.6(iii).

Consequently, $a \in \text{corona}(G) \cup N(\text{core}(G))$. In other words, we get that

$$V(G) \subseteq \text{corona}(G) \cup N(\text{core}(G)),$$

as required.

As for the second equality, let us notice that $V(C) \subseteq \text{corona}(G)$, by *Case 1*. If $a \in \text{corona}(G) - V(C)$, then by Theorem 1.6(ii), there are $S \in \Omega(G)$ and $b \in N_1(C)$, such that $a \in S \cap V(T_b) \in \Omega(T_b)$. Hence $a \in \text{corona}(T_x)$, and therefore,

$$\text{corona}(G) - V(C) \subseteq \cup \{\text{corona}(T_x) : x \in N_1(C)\}.$$

Theorem 1.6(i) assures that $\cup \{\text{corona}(T_x) : x \in N_1(C)\} \subseteq \text{corona}(G)$. In conclusion, $\text{corona}(G) = V(C) \cup (\cup \{\text{corona}(T_x) : x \in N_1(C)\})$. ■

The graph G_2 from Figure 4 shows that the equality $|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G)$ is not true for unicyclic non-König-Egerváry graphs.

Theorem 2.4 *If G is a unicyclic graph, then*

$$2\alpha(G) \leq |\text{corona}(G)| + |\text{core}(G)| \leq 2\alpha(G) + 1.$$

Moreover, G is a non-König-Egerváry graph if and only if

$$|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G) + 1.$$

Proof. By Theorem 2.2(ii), the equality $2\alpha(G) = |\text{corona}(G)| + |\text{core}(G)|$ holds for every unicyclic König-Egerváry graph.

Assume now that G is not a König-Egerváry graph.

Let $S \in \Omega(G)$. According to Theorem 2.3 and Lemma 1.5(i), we infer that

$$|S| + |\text{corona}(G) - S| + |N(\text{core}(G))| = |V(G)| = \alpha(G) + \mu(G) + 1,$$

which implies $|\text{corona}(G) - S| + |N(\text{core}(G))| = \mu(G) + 1$.

By Theorem 1.1, there is a matching M_1 from $S - \text{core}(G)$ into $\text{corona}(G) - S$, which implies $|S - \text{core}(G)| \leq |\text{corona}(G) - S|$.

Lemma 1.5 implies that there is a matching M from $N(\text{core}(G))$ into $\text{core}(G)$, that can be enlarged to a maximum matching, say M_2 , of G .

Since M_2 matches $\mu(G)$ vertices from $A = (\text{corona}(G) - S) \cup (N(\text{core}(G)))$ by means of $\mu(G)$ edges, and $|A| = \mu(G) + 1$, it follows that $M_2 - M$ matches $|\text{corona}(G) - S| - 1$ vertices from A into $S - \text{core}(G)$, because M saturates $N(\text{core}(G))$ and no edge joins a vertex of $\text{core}(G)$ to some vertex from $\text{corona}(G) - S$. Hence, taking into account that $M \cup M_1$ is a matching of G , while M_2 is a maximum matching, we obtain

$$\begin{aligned} \mu(G) &= |M_2| = |N(\text{core}(G))| + |\text{corona}(G) - S| - 1 \leq \\ &\leq |N(\text{core}(G))| + |S - \text{core}(G)| = |M| + |M_1| \leq \mu(G), \end{aligned}$$

which implies $|S - \text{core}(G)| = |\text{corona}(G) - S| - 1$.

Finally, we infer that

$$\begin{aligned} |\text{corona}(G)| &= |S \cup (\text{corona}(G) - S)| = \alpha(G) + |\text{corona}(G) - S| = \\ &= \alpha(G) + |S - \text{core}(G)| + 1 = 2\alpha(G) - |\text{core}(G)| + 1, \end{aligned}$$

and this completes the proof. ■

Theorem 2.5 *If G is a unicyclic non-König-Egerváry graph, then*

$$\ker(G) = \cup \{\ker(T_x) : x \in N_1(C)\} = \text{core}(G).$$

Proof. Since T_x is bipartite, by Theorem 1.3(ii) implies that $\ker(T_x) = \text{core}(T_x)$, for every $x \in N_1(C)$.

According to Theorems 1.3(i) and 1.6(iii), it follows that

$$\ker(G) \subseteq \text{core}(G) = \cup \{\text{core}(T_x) : x \in N_1(C)\}.$$

Hence $A_x = \ker(G) \cap V(T_x) \subseteq \text{core}(G) \cap V(T_x) = \text{core}(T_x) = \ker(T_x)$, for every $x \in N_1(C)$.

Assume that $A_q \neq \ker(T_q)$ for some $q \in N_1(C)$. It follows that $d(A_q) < d(\ker(T_q))$.

Since, by Theorem 1.6(i), we have $\ker(G) \subseteq \text{core}(G)$, Lemma 1.5(ii) ensures that $N[V(C)] \cap \ker(G) = \emptyset$. Consequently, the set $W = (\ker(G) - A_q) \cup \ker(T_q)$ is independent, and satisfies

$$\begin{aligned} d(\ker(G)) &= d(\cup \{\ker(G) \cap V(T_x) : x \in N_1(C)\}) = \\ &= \sum_{x \in N_1(C)} d(\ker(G) \cap V(T_x)) = \sum_{x \in N_1(C)} d(A_x) = d(A_q) + \sum_{x \in N_1(C) - \{q\}} d(A_x) < \\ &< d(\ker(T_q)) + \sum_{x \in N_1(C) - \{q\}} d(A_x) = d(W) \leq \max \{d(X) : X \subseteq V(G)\} = d(\ker(G)), \end{aligned}$$

which is a contradiction.

Therefore, we infer that

$$\ker(G) \cap V(T_x) = \text{core}(G) \cap V(T_x) = \text{core}(T_x) = \ker(T_x)$$

hold for each $x \in N_1(C)$. Hence,

$$\ker(G) = \text{core}(G) = \cup \{\text{core}(T_x) : x \in N_1(C)\} = \cup \{\ker(T_x) : x \in N_1(C)\},$$

as claimed. ■

Remark 2.6 *If G is a unicyclic König-Egerváry graph that is non-bipartite, then the difference between $|\text{core}(G)|$ and $|\ker(G)|$ may equal any non-negative integer. For instance, the graph G_{2k+1} from Figure 5 satisfies $\alpha(G_{2k+1}) = k + 3$, $\mu(G_{2k+1}) = k + 1$, while $|\text{core}(G_{2k+1})| - |\ker(G_{2k+1})| = k - 1, k \geq 1$.*

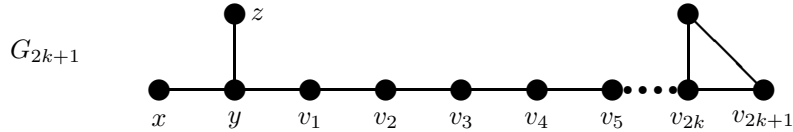


Figure 5: $\ker(G_{2k+1}) = \{x, z\}$, while $\text{core}(G_{2k+1}) = \{x, z, v_1, v_3, \dots, v_{2k-1}\}$.

3 Conclusions

The equality $\text{core}(G) = \ker(G)$ may fail for some non-bipartite unicyclic König-Egerváry graphs; e.g., the graphs G_1 and G_2 from Figure 3 satisfy $\ker(G_1) = \{a, b\} \neq \text{core}(G_1) = \{a, b, c\}$, while $\ker(G_2) = \text{core}(G_2) = \{x, y, z\}$.

Problem 3.1 *Characterize non-bipartite unicyclic König-Egerváry graphs G satisfying $\text{core}(G) = \ker(G)$.*

The non-unicyclic graphs G_1 and G_2 from Figure 6 satisfy $|\text{corona}(G_1)| + |\text{core}(G_1)| = 2\alpha(G_1)$ and $|\text{corona}(G_2)| + |\text{core}(G_2)| = 2\alpha(G_2) + 1$.

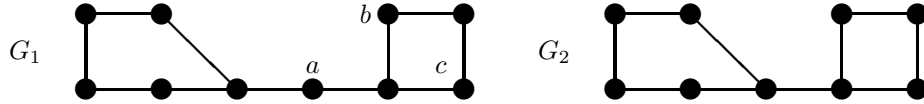


Figure 6: $\text{core}(G_1) = \{a, b, c\}$ and $\text{core}(G_2) = \emptyset$.

Problem 3.2 *Characterize graphs satisfying*

$$2\alpha(G) \leq |\text{corona}(G)| + |\text{core}(G)| \leq 2\alpha(G) + 1.$$

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